DSGE and VAR models

Ed Herbst[1]

[1] Federal Reserve Board

October 23, 2018

DSGE Models and VARs

DSGE Model

- Fully articled general equilibrium model
- Observed and unobserved variables
- parameters θ
- Has following state space representation (if linear + normal shocks)

$$s_t = T(\theta)s_{t-1} + R(\theta)\epsilon_t, \quad \epsilon_t \sim N(0, Q) \tag{1}$$
$$y_t = D(\theta) + Z(\theta)s_t + \eta_t, \quad \eta_t \sim N(0, H) \tag{2}$$

Tight link to theory can be a blessing and curse...

What's the link to a more flexible class of models we studied: Vector Autoregressions?

Well...

Leeper, Walker, and Yang

RBC model with taxes

$$\frac{1}{C_t} = \alpha\beta E_t\left[(1-\tau_{t+1})\frac{1}{C_{t+1}}\frac{Y_{t+1}}{K_t}\right] \quad (3)$$

$$A_t K_{t-1}^{\alpha} = C_t + K_t = Y_t \quad (4)$$

$$k_t = \log(K_t) - \log(K_{ss}), \hat{\tau}_t = \log(\tau_t) - \log(\tau_{ss}), \ldots$$

Log linearized:

$$E_t[k_{t+1}] - (\theta^{-1}+\alpha)k_t + \alpha\theta^{-1}k_{t-1} = E_t\left[a_{t+1} - \theta^{-1}a_t\right] + \left\{\theta^{-1}(1-\theta)\left(\frac{\tau}{1-\tau}\right)\right\} E_t[\tau_{t+1}$$

If a_t is iid,

$$k_t = \alpha k_{t-1} + a_t - (1-\theta)\left(\frac{\tau}{1-\tau}\right)\sum_{i=0}^{\infty}\theta^i E_t\hat{\tau}_{t+i+1}$$

$\theta = \alpha\beta(1-\tau_{ss})$.

Fiscal Foresight

Imagine taxes are known q periods in advance $\tau_t = \tau_{ss}\exp(\epsilon_{\tau,t-q})$.
Makes sense because of legislative lags, et cetera,
$q = 0$ implies:

$$k_t = \alpha k_{t-1} + \epsilon_{a,t}$$

$q = 1$ implies:

$$k_t = \alpha k_{t-1} + \epsilon_{a,t} - \kappa\epsilon_{\tau,t}$$

$q = 2$ implies:

$$k_t = \alpha k_{t-1} + \epsilon_{a,t} - \kappa(\epsilon_{\tau,t-1} + \theta\epsilon_{\tau,t})$$

$q = 2$ implies:

$$k_t = \alpha k_{t-1} + \epsilon_{a,t} - \kappa(\epsilon_{\tau,t} + \theta\epsilon_{\tau,t-1} + \theta^2\epsilon_{\tau,t})$$

$\kappa = (1-\theta)(\tau/(1-\tau))$

Comment

- $q = 0$, implies iid tax shocks have no effect on capital accumulation. (standard result.)
- $q > 0$ implies agents adjust capital contemporaneously, even serially uncorrelated tax hikes reduce capital consumption.
- Fiscal foresight: moving average terms in equilibrium representation.
- *More* recent new is discounted by $\theta = \alpha\beta(1 - \tau) < 1$ relative to older news!
- Tax *rates* still discounted in the usual way.

Econometrics

Set $a_t = 0$,

$$(1 - \alpha L)k_t = -\kappa(L + \theta)\epsilon_{\tau,t}$$

inverting this requires

$$\frac{1 - \alpha L}{1 + \theta^{-1} L} k_t$$

to be a convergent sequence.

But this is only true is $|\theta| > 1$

$\implies \{\epsilon_{\tau,t-j}\}_{j=0}^{\infty}$ is not fundamental for $\{k_{t-j}\}_{j=0}^{\infty}$.

What does the AR representation for k_t look like?

More econometrics

Derive Wold representation for k_t, determine one step ahead forecast errors.

Quick trick: **Blaschke Factor** [Lippi and Reichlin (1994)] flip root of MA process from inside to outside the unit circle (same ACF) using $[(L+\theta)/(1+\theta L)]$:

$$
\begin{aligned}
(1-\alpha L)k_t &= -k(L+\theta)\left[\frac{1+\theta L}{L+\theta}\right]\left[\frac{L+\theta}{1+\theta L}\right]\epsilon_{\tau,t} && (5)\\
&= -\kappa(1+\theta L)\epsilon^*_{\tau,t} && (6)\\
&= -\kappa(\theta\epsilon^*_{\tau,t-1}+\epsilon^*_{\tau,t}). && (7)
\end{aligned}
$$

- By observing current and past capital, econometrician recovers current and past ϵ^*_τ, not ϵ_{tau}
- The econometricians innovatives are the statistical shock associated with estimated the autoregressive representation
- This shocks shocks represent information that is mostly “old news” to the agents of the economy.

Relationship between fundamental shocks and econometricians

$$\epsilon^*_{\tau,t} = \left[\frac{L+\theta}{1+\theta L}\right]\epsilon_{\tau,t} = (L+\theta)\sum_{j=0}^{\infty} -\theta^j \epsilon_{\tau,t-j} \tag{8}$$

$$= \theta\epsilon_{\tau,t} + (1-\theta^2)\epsilon_{\tau,t-1} - \theta(1-\theta^2)\epsilon_{\tau,t-2} + \theta^2(1-\theta^2)\epsilon_{\tau,t-3} \tag{9}$$

- An econometrican who ignores foresight will discount the innovations incorrectly.
- Econometrica, yesterdays innovations has *less* effect than today's innovation.
- Agents discount news in the other way.

This causes big problems.

- By not modelling forecast, the econometrician has a smaller information set.
- The extent to which private agents condition on information that is not captured by current and past variables in the econometricians information set determines the error in the VAR.

- We can map this into θ directly.

For the agents:

$$
\begin{aligned}
E[(k_{t+1} - E[k_{t+1}|\epsilon^t])^2] &= E\left[\left(-\frac{\kappa(L+\theta)}{1-\alpha L}\epsilon_{\tau,t+1} - L^{-1}\frac{\kappa(L+\theta)}{1-\alpha L} + \kappa\theta]\epsilon_{\tau,t}\right)^2\right] \\
&= (\kappa\theta)^2\sigma_\tau^2.
\end{aligned}
$$

For the econometrician:

$$
\begin{aligned}
E[(k_{t+1} - E[k_{t+1}|k^t])^2] &= E\left[\left(-\frac{\kappa(L+\theta)}{1-\alpha L}\epsilon_{\tau,t+1} - L^{-1}\frac{\kappa(L+\theta)}{1-\alpha L} + \kappa]\frac{L+\theta}{1+\theta L}\epsilon_{\tau,t}\right)^2\right. \\
&= (\kappa)^2\sigma_\tau^2.
\end{aligned}
$$

Impulse Response Functions

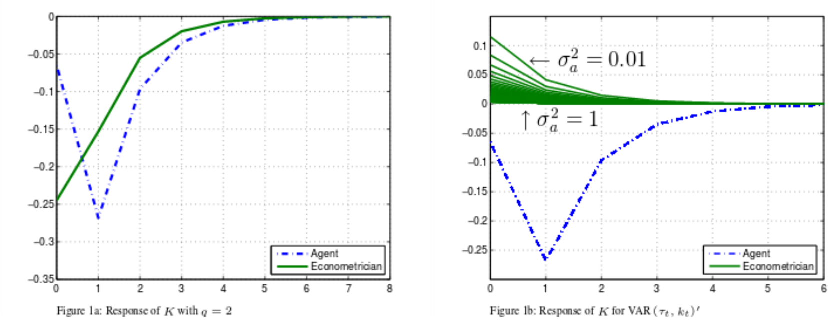


Figure 1. Responses of Capital to Tax Increase with $\alpha = 0.36, \beta = 0.99, \tau = 0.25$. Figure 1a plots the response of (13) and (14). Figure 1b plots the response to the VAR $(\tau_t, k_t)'$. Both figures assume two quarters of foresight.

General Analysis

Let's simplify

- Number of observables (y_t) is equal to the number of structural shocks (ϵ_t).
- $\eta_t = 0$
- $D(\theta) = 0$.

Question: Can we write DSGE model as a VAR? Let's write the VAR slightly differently:

$$\begin{aligned} s_t &= As_{t-1} + B\epsilon_t & (10) \\ y_t &= Cs_{t-1} + D\epsilon_t & (11) \end{aligned}$$

$A = T, B = R, C = ZT, D = ZR$
This means that

$$\epsilon_t = (D)^{-1}(y_t - Cs_{t-1}).$$

Using the state equation

$$s_t = (A - BD^{-1}C)s_{t-1} + BD^{-1}y_t.$$

Solving backwards,

$$s_t = (A - BD^{-1}C)^{t-1} s_0 + \sum_{j=0} (A - BD^{-1}C)^{j-1} BD^{-1} y_{t-j}$$

If eigenvalues of $(A - BD^{-1}C)$ are less than one in modulus, then

$$lim_{t\to\infty}(A - BD^{-1}C)^t \to 0$$

And we can write the states as a combination of the history of observations. So

$$y_t \approx C \sum_{j=0}^{\infty} (A - BD^{-1}C)^{j-1} BD^{-1} y_{t-1-j} + D\epsilon_t.$$

We have a VAR(∞) representation for y_t whose innovations coincide with the structural shocks of our DSGE model!

The condition that *eigenvalues of* $(A - B(D)^{-1}C)$ *are less than one in modulus* is known as the **Poor Man's Invertibility Condition.** [Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007).]

A Small Example

Consider the **permanent income consumption model** [Sargent (1987)]

$$
\begin{aligned}
c_{t+1} &= c_t + \sigma_w (1 - R^{-1}) w_{t+1} \\
y_{t+1} - c_{t+1} &= -c_t + \sigma_w R^{-1} w_{t+1}
\end{aligned}
\qquad (12)\ (13)
$$

$y_{t+1} = \sigma_w w_{t+1}$ is an i.i.d labor income process

$R > 1$ is a constant gross interest rate on financial assets

c_t in the unobserved state

$y_t - c_t$ is observed by the econometrician.

Calibration: $R = 1.2$ and $\sigma_w = 1$.

A simple model

Consider the **permanent income consumption model** [Sargent (1987)]

$$\begin{aligned} c_{t+1} &= c_t + \sigma_w (1 - R^{-1}) w_{t+1} \\ y_{t+1} - c_{t+1} &= -c_t + \sigma_w R^{-1} w_{t+1} \end{aligned} \tag{14}$$

(15)

$y_{t+1} = \sigma_w w_{t+1}$ is an i.i.d labor income process

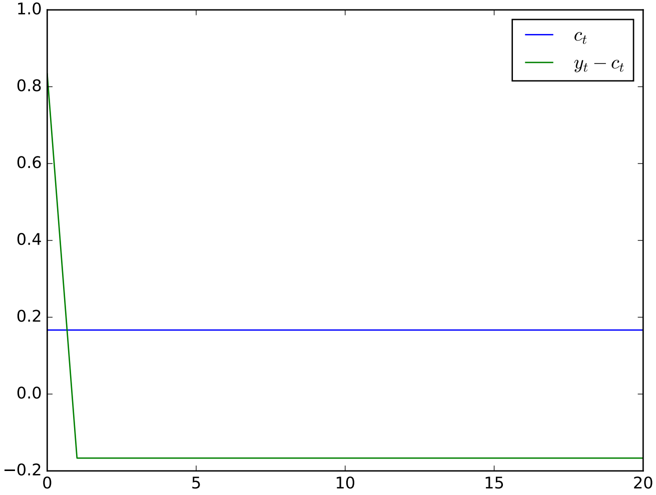
$R > 1$ is a constant gross interest rate on financial assets

c_t in the unobserved state

$y_t - c_t$ is observed by the econometrician.

Calibration: $R = 1.2$ and $\sigma_w = 1$.

Impulse Response to w_t



Can we recover this in a VAR?

Let's write it in our notation:
state equation: $s_t = [c_t, y_t - c_t]$

$$T(\theta) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad R(\theta) = \begin{bmatrix} \sigma_w(1 - R^{-1}) \\ \sigma_w R^{-1} \end{bmatrix}, \quad Q(\theta) = 1.0$$

observable equation:

$$DD(\theta) = 0, \quad ZZ(\theta) = [0, 1], \quad HH(\theta) = 0$$

Poor Man's invertibility condition: $T - R(ZR)^{-1}ZT$

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \sigma_w(1 - R^{-1}) \\ \sigma_w R^{-1} \end{bmatrix} \frac{R}{\sigma_w} [-1, 0] = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$$

The maximum eigenvalue of this matrix $= R > 1$! Poor man's invertibility condition doesn't hold.

Upshot

- If the invertibility condition is met: With enough data, VAR forecast errors $D\epsilon_t \implies$ we can recover the structural shocks

- If not, we still have an VAR representation in the observables, but the VAR innoations not longer correspond to linear combinations of the structural shocks.

- What's the issue: when the invertibility condition is not the observables do not perfectly reveal the state vector.

Some Analysis

The **innovations representation** of the state space system

$$\hat{s}_t = \underbrace{T}_{A}\hat{s}_{t-1} + \underbrace{TP_tZF_t^{-1}}_{\hat{B}_t}u_t \tag{16}$$

$$y_t = \underbrace{ZT}_{C}\hat{s}_{t-1} + \underbrace{ZTP_tZF_t^{-1}}_{\hat{D}_t}u_t \tag{17}$$

Initialization: $s_0 \sim (\hat{s}_0, \Sigma_0)$

Updated state: $\hat{s}_t = E[s_t|\{y_j\}_{j=1}^{t}]$

Forecast error $y_t - E[y_t|\{y_j\}_{j=1}^{t-1}] = D_t u_t$

For general conditions: $\hat{B}_t \to \hat{B}, \hat{D}_t \to \hat{D}$

After *a lot* of algebra, you can show that $\hat{D}\hat{D}' = DD' + C\Sigma C'$ where Σ is the long run variance associated with $s_t|y_t, \ldots$.
If $A - BD^{-1}C$ has eigenvalues less than unity in modulus, $\Sigma = 0$.

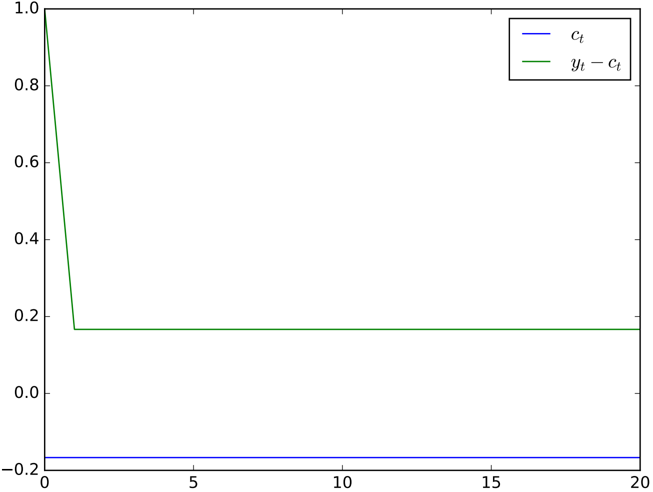
Innovations Representation of Simple Model

$$
\begin{aligned}
c_{t+1} &= c_t + \sigma_w (R^{-1} - 1) u_{t+1} \\
y_{t+1} - c_{t+1} &= -c_t + \sigma_w u_{t+1}
\end{aligned}
\tag{18, 19}
$$

In this framework, the ABCD form yields $A - BD^{-1}C = \frac{1}{R} < 1$

What does a shock to u_t look like?

Impulse Response to u_t



Upshot

- Consumption response is negativ!
- Forecast errors in $y_t - c_t$ arise from shocks to income, w_t or from errors in estimating past consumption.
- The Kalman filter optimally allocates ϵ_t to these two sources.
- Recall invertibility is property of the observables.

What if We observed y_{t+1} instead of $y_{t+1} - c_{t+1}$

?

How to deal with this problem in general?

- Expand observable vector (see above). [Ramey (2011)]
- Use a factor model setup [Forni and Gambetti (2010)]
- Estimate the DSGE Model Directly.
- Use theory to tell us abou the cruz of non fundamentalness [Ravn and Mertens (2010)]