Econ 616: Problem Set 1

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Problem 1

Let $\phi(z) \equiv 1 - \phi_1 z - \phi_2 z^2$. What we need to show is the solution of the equation $\phi(z) = 0$ lies outside of unit circle. Let z_1 and z_2 be the solutions of $\phi(z) = 0$.

- **Case 1:** Suppose $\phi_1^2 + 4\phi_2 \leq 0$. Then we have either $z_1 = z_2$ or that z_1 and z_2 are complex numbers and conjugate of each other. In any case the norm of the solution is given by $\sqrt{\left|\frac{1}{\phi_2}\right|}$. Hence the condition is $|\phi_2| < 1$.
- **Case 2:** Suppose $\phi_1^2 + 4\phi_2 > 0$. Now both solutions are real number. Suppose $\phi_2 = 0$. This is AR(1) model and the condition is $|\phi_1| < 1$. Suppose $\phi_2 \neq 0$. It would be easier to analyze the equation $\psi(z) = 0$ where $\psi(z) \equiv z^2 + \frac{\phi_1}{\phi_2} z - \frac{1}{\phi_2}$ which has the same solutions as $\phi(z) = 0$. If $\phi_2 < 0$, then the conditions are $\psi(1) > 0$ and $\psi(-1) > 0$ which means that $\phi_2 + \phi_1 < 1$ and $\phi_2 - \phi_1 < 1$. If \$$\phi_2 > 0$ \$, then the conditions are $\psi(1) < 0$ and $\psi(-1) < 0$ which means that $\phi_2 + \phi_1 < 1$ and $\phi_2 - \phi_1 < 1$.

Combining all, we have

$$\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1 \text{ and } \phi_2 > -1.$$

Problem 2

Multiplying both sides of the equation by y_{t-j}, we have

$$y_t y_{t-j} = \sum_{i=1}^{p} \phi_i y_{t-i} y_{t-j} + \epsilon_t y_{t-j}.$$

Taking the expectation, we have

$$E(y_t y_{t-j}) = \sum_{i=1}^{p} \phi_i E(y_{t-i} y_{t-j}) + E(\epsilon_t y_{t-j}).$$

Or we can rewrite the above as

$$\gamma_{yy,j} = \sum_{i=1}^{p} \phi_i \gamma_{yy,|i-j|} + E(\epsilon_t y_{t-j}).$$

For $j = 0$, $E(\epsilon_t y_{t-j}) = E(\epsilon_t y_t) = E(\epsilon_t^2) = \sigma_\epsilon^2$ which gives the first equation. For $j = 1, \ldots, p$, $E(\epsilon_t y_{t-j}) = 0$ which gives the rest.

For the AR(3) process in Problem 2, we have

$$\begin{aligned}
1 &= \gamma_{yy,0} - (1.3\gamma_{yy,1} - 0.9\gamma_{yy,2} + 0.3\gamma_{yy,3}) \\
0 &= 1.3\gamma_{yy,0} - \gamma_{yy,1} + (-0.9\gamma_{yy,1} + 0.3\gamma_{yy,2}) \\
0 &= -0.9\gamma_{yy,0} - \gamma_{yy,2} + (1.3\gamma_{yy,1} + 0.3\gamma_{yy,1}) \\
0 &= 0.3\gamma_{yy,0} - \gamma_{yy,3} + (1.3\gamma_{yy,2} - 0.9\gamma_{yy,1})
\end{aligned}$$

Solutions to this system of equations are

$\gamma_{yy,0} = 3.38$ $\gamma_{yy,1} = 2.45$ $\gamma_{yy,2} = 0.88$ $\gamma_{yy,3} = -0.48$

Problem 3

See jupyter notebook

Problem 4

- The least squares estimates can be written as:

$$\hat{\rho}_{LS} = \rho + \left(\sum_{t=2}^{T} y_{t-1}^2 \right)^{-1} \sum_{t=2}^{T} \epsilon_t y_{t-1} \tag{1}$$

  Consider an alternative representation of y_t.

$$\begin{aligned}
t \text{ even}: &\quad y_t = y_t^e = \rho^2 y_t^e + \sigma\epsilon_t + \rho\alpha\sigma\epsilon_{t-1} \\
t \text{ odd}: &\quad y_t = y_t^o = \rho^2 y_t^o + \alpha\sigma\epsilon_t + \rho\sigma\epsilon_{t-1}.
\end{aligned} \tag{2}$$

  Consider,

$$\begin{aligned}
\frac{1}{T}\sum_{t=2}^{T} y_{t-1}^2 &\approx \frac{1}{2}\frac{1}{T/2}\sum_{t=2}^{T/2}(y_{2(t-1)+1}^o)^2 + \frac{1}{2}\frac{1}{T/2}\sum_{t=2}^{T/2}(y_{2t}^e)^2 \\
&\to \frac{1}{2}\frac{\alpha^2+\rho^2}{1-\rho^4}\sigma^2 + \frac{1}{2}\frac{1+\alpha^2\rho^2}{1-\rho^4}\sigma^2. \\
&= \frac{1}{2}\frac{(1+\alpha^2)(1+\rho^2)}{(1-\rho^2)(1+\rho^2)}\sigma^2. \\
&= \frac{1+\alpha^2}{2}\frac{\sigma^2}{1-\rho^2}.
\end{aligned} \tag{3}$$

Moreover, The sequence $\{\epsilon_t y_{t-1}\}$ is a Martingale difference sequence (MDS). If $|\rho| < 1$, $\frac{1}{T}\sum_{i=1}^{T} \epsilon_t y_{t-1} \to 0$ as $T \to \infty$. Thus, the least squares estimator is consistent.

- Using arguments along the lines of (3) and the central limit theorem for MDS yields the asymptotic distribution for $\hat{\rho}_{LS}$. In particular, The least squares estimator $\hat{\rho}_{LS} = \left(\sum_{t=2}^{T} y_{t-1}^2\right)^{-1} \sum_{t=2}^{T} y_t y_{t-}$ in large samples behaves such that

$$\sqrt{T}(\hat{\rho}_{LS} - \rho) \sim N(0, \mathbb{V}_{\hat{\rho}}). \quad (4)$$

This variance take the form:

$$\mathbb{V}_{\hat{\rho}} = \frac{\mathbb{E}[\epsilon_t^2 y_{t-1}^2]}{\mathbb{E}[y_{t-1}^2]^2} \quad (5)$$

Tedious algebra yields:

$$E[y_{t-1}^2] = \frac{1+\alpha^2}{2}\frac{\sigma^2}{1-\rho^2} \quad (6)$$

$$E[\epsilon_t^2 y_{t-1}^2] = \frac{\alpha^2(1+\alpha^2\rho^2) + (\alpha^2+\rho^2)}{2}\frac{\sigma^4}{1-\rho^4} \quad (7)$$

Thus:

$$\mathbb{V}_{\hat{\rho}} = 2\left(\frac{\rho^2 + 2\alpha^2 + \alpha^4\rho^2}{1 + 2\alpha^2 + \alpha^4}\right)\left(\frac{1-\rho^2}{1+\rho^2}\right) \quad (8)$$

- It is easy to see that this quantity converges in probability to

$$\mathbb{V}_{\hat{\rho}}^* = \frac{\mathbb{E}[\epsilon_t^2]}{\mathbb{E}[y_{t-1}^2]} = 1 - \rho^2 \quad (9)$$

- Tedious algebra shows that:

$$\mathbb{V}_{\hat{\rho}} \leq \mathbb{V}_{\hat{\rho}}^*.$$

Thus, the typical estimator for standard errors is inconsistent and in particular it overstates the variance of $\hat{\rho}_{LS}$.

Problem 5

Recall from the lecture notes that the HP filter can be written as:

$$f^{HP}(\omega) = \left[\frac{16\sin^4(\omega/2)}{1/1600 + 16\sin^4(\omega/2)}\right]^2. \quad (10)$$

The spectrum for the AR1 can be written as:

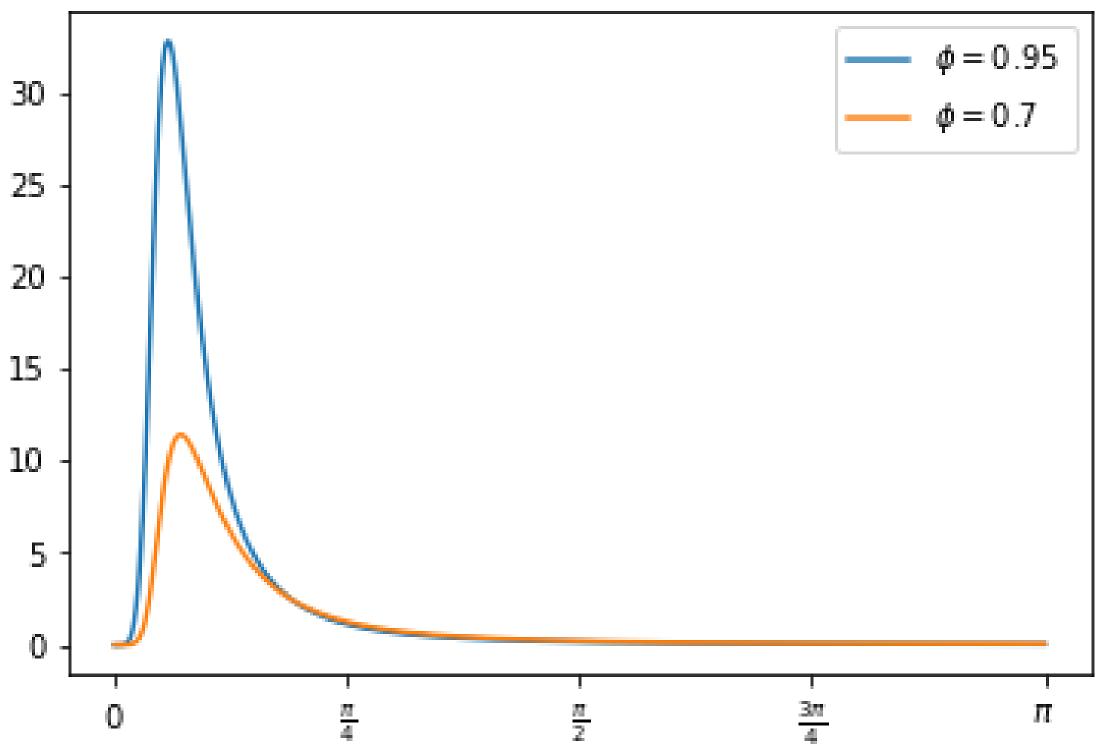
$$f^{AR}(\omega) = \left[1 - 2\phi\cos\omega + \phi^2(\cos\omega^2 + \sin^2\omega)\right]^{-1}. \quad (11)$$

From the lecture notes, we know that:

$$f^Y(\omega) = f^{HP}(\omega)f^{AR}(\omega) \tag{12}$$

The spectrum for $\phi = 0.95$ and $\phi = 0.70$ is plotted in Figure 3. The spectrum peaks at about $\pi/8$, which is associated with a cycle lasting about 16 quarters = $(2\pi/(\pi/8))$. As ϕ increases, this peak sharpens. So here the HP filter is introducing as spurious periodicity in our data.

```
<ipython-input-14-04db9a7f9b5a>:6: RuntimeWarning: divide by zero encountered in divide
  return ( (sigma**2/(2*omega)) /
<ipython-input-14-04db9a7f9b5a>:9: RuntimeWarning: invalid value encountered in multiply
  f = lambda omega, **kwds: f_HP(omega)*f_AR1(omega, **kwds)
<matplotlib.legend.Legend at 0x7f5cd9dfb8b0>
```



Another way to see this is look at the autocovariance function of the implied process, which we can recover by the inverse fourier transform as discussed in class:

$$\gamma_k = \int_{-\pi}^{\pi} f^Y(\omega)e^{i\omega k}. \tag{13}$$

The HP filter induces complex dynamics into the process!

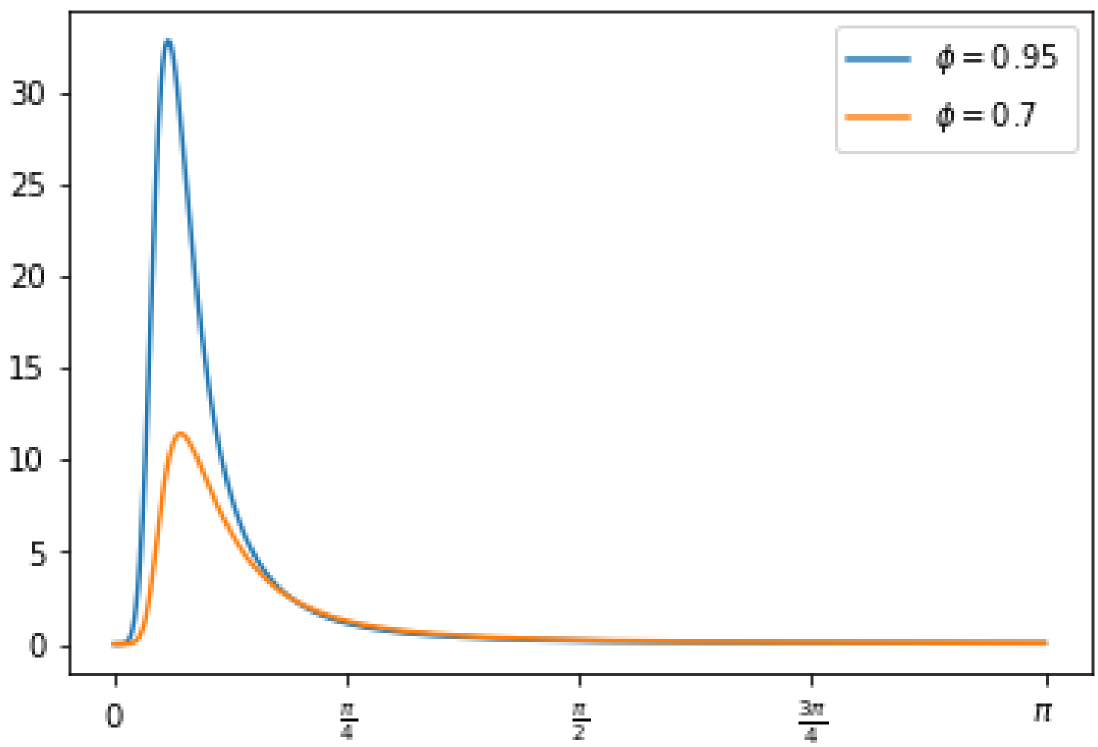


Figure 1: Spectrum Associated with HP filtering an AR(1)

```
from scipy.integrate import quad
H = 20
really_small = 1e-8
acf = [quad(lambda omega:
                 2*f(omega)*np.cos(omega*k), really_small, np.pi)[0]
            for k in np.arange(H)]
plt.plot(acf)

phi = 0.95
acf = [ phi**j / (1-phi**2) for j in np.arange(H)]
plt.plot(acf, linestyle='dashed')
```

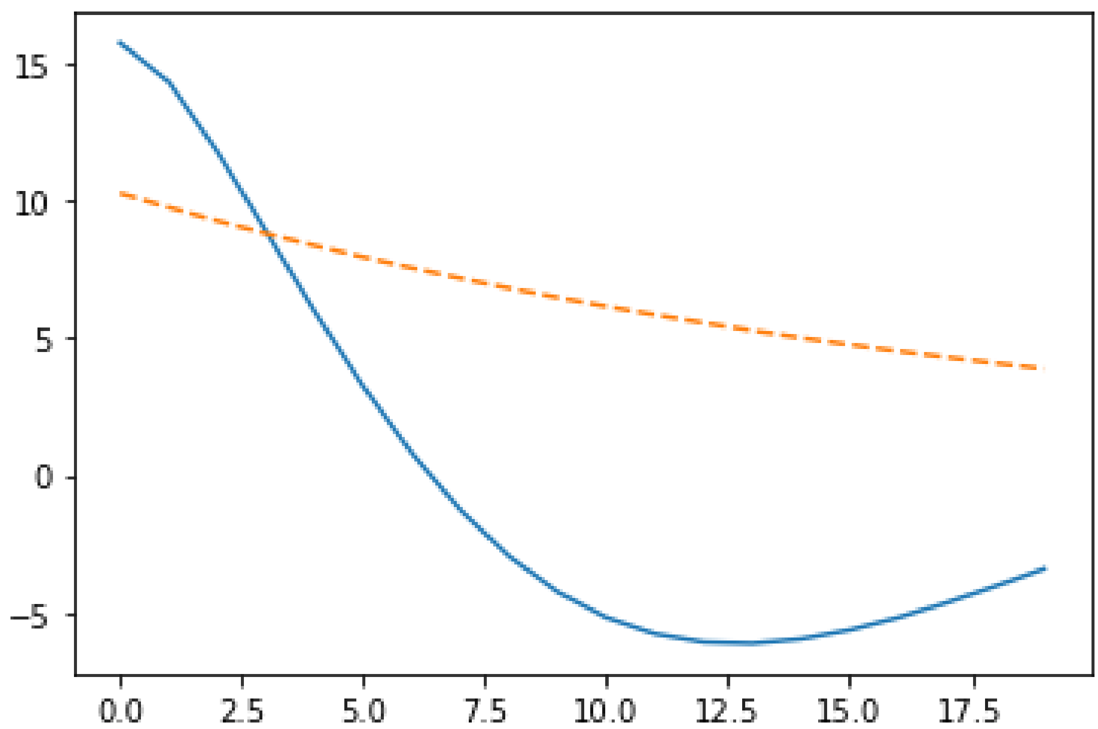


Figure 2: ACF of AR(1) vs. ACF of HP filtered Component