Econ 616: Problem Set 2

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Problem 1

Consider the following AR(1) process, initialized in the infinite past:

$$y_t = \theta y_{t-1} + \epsilon_t, \tag{1}$$

where $\epsilon_t \sim iid\mathcal{N}(0,1)$.

1. Suppose you have a sample of observations $Y^T = \{y_0, y_1, \ldots, y_T\}$. Derive the conditional likelihood function $p(Y^T|\theta, y_0)$ for θ based on Y^T.

2. Consider the following prior for θ: $\theta \sim \mathcal{N}(0, \tau^2)$. Show that the posterior distribution of θ is of the form

$$\theta|Y^T \sim \mathcal{N}(\tilde{\theta}_T, \tilde{V}_T), \tag{2}$$

where

$$\tilde{\theta}_T = \left(\sum y_{t-1}^2 + \tau^{-2}\right)^{-1} \sum y_t y_{t-1} \tag{3}$$

$$\tilde{V}_T = \left(\sum y_{t-1}^2 + \tau^{-2}\right)^{-1} \tag{4}$$

3. Suppose the goal is to forecast y_{T+2} based on information up until time T, given by the sample Y^T. Show that under the loss function

$$L(y_{T+2}, \hat{y}_{T+2T}) = (y_{T+2} - \hat{y}_{T+2|T})^2 \tag{5}$$

where y_{T+2} is the actual value and $\hat{y}_{T+2|T}$ is the predicted value, the optimal (minimizing posterior expected loss) forecast is given by

$$\hat{y}^{opt}_{T+2|T} = E[y_{T+2}|Y^T]. \tag{6}$$

4. Using the results from (ii), calculate the optimal two-step ahead predictor for the estimated AR(1) model. Notice that

$$E[y_{T+2}|Y^T] = \int [y_{T+2}|\theta, Y^T]p(\theta|Y^T)d\theta. \tag{7}$$

5. Suppose that data are generated from an AR(2) model

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t. \tag{8}$$

but the Bayesian bases his/her analysis on an AR(1) model. What happens to the mean and variance of the posterior distribution in (iv) as $T \longrightarrow \infty$.

Problem 2

Consider the following two models for the time series $Y^T = \{y_1, \ldots, y_T\}$:

$$\mathcal{M}_0 \quad : \quad y_t = u_t, \quad u_t \sim iid\mathcal{N}(0,1), \tag{9}$$

$$\mathcal{M}_1 \quad : \quad y_t = \theta y_{t-1} + u_t, \quad u_t \sim iid\mathcal{N}(0,1). \tag{10}$$

You may assume that $y_0 = 0$.

1. Derive the joint probability density function (pdf) for Y^T conditional on the initial observation and the model parameters for $\mathcal{M}_0$ and $\mathcal{M}_1$.

2. Define the likelihood ratio statistic

$$LR_T = 2\ln \frac{\max_{\theta\in\Theta}\ p(Y^T|\theta, \mathcal{M}_1)}{p(Y^T|\mathcal{M}_0)}, \tag{11}$$

where $p(Y^T|\mathcal{M}_0)$ and $p(Y^T|\theta, \mathcal{M}_1)$ denote the pdf's derived in (i). Derive the limit distribution of LR_T under the assumption that data have been generated from $\mathcal{M}_0$.

Now consider the following prior distribution for θ in $\mathcal{M}_1$: $\theta \sim \mathcal{N}(0, \tau^2)$.

1. Derive the posterior distribution of θ under conditional on $\mathcal{M}_1$.

2. Derive the marginal data density for model $\mathcal{M}_1$

$$p(Y^T|\mathcal{M}_1) = \int p(Y^T|\theta, \mathcal{M}_1)p(\theta)d\theta. \tag{12}$$

3. Suppose the prior probabilities for models $\mathcal{M}_0$ and $\mathcal{M}_1$ are equal to 0.5. Find an expression in terms of $y_1, \ldots, y_T$ for the log posterior odds of $\mathcal{M}_1$ versus $\mathcal{M}_0$:

$$LPO_T = \ln \frac{\{\mathcal{M}_1|Y^T\}}{\{\mathcal{M}_0|Y^T\}}.$$

4. Suppose that Y^T has been generated from $\mathcal{M}_0$. What happens to to LPO_T as $T \longrightarrow \infty$. Compare the asymptotic behavior of LPO_T and LR_T and discuss some of the differences between Bayesian and classical testing.

Problem 3

This problem set draws on a few influential papers about the effects of fiscal policy.

- Blanchard, O., & Perotti, R. (2002). An empirical characterization of the dynamic effects of changes in government spending and taxes on output. *The Quarterly Journal of Economics*, 117(4), 1329–1368. `http://dx.doi.org/10.1162/003355302320935043`

- Mertens, K., & Ravn, M. O. (2014). A reconciliation of svar and narrative estimates of tax multipliers. *Journal of Monetary Economics*, 68, 1–19. `http://dx.doi.org/10.1016/j.jmoneco.2013.04.004`

Our goal is to assess the effects of an unanticipated change in taxes on aggregate output. To do this, we're going to use a structural VAR, identified in two ways. I have provided skeleton code for this exercise using either Matlab (see attached files) or other programming language.

Get the data, detrend it

Load the data file `mertens_ravn.csv`. The first column refers to the (quarterly) date, while second through fourth column refers to the log of tax revenues (T_t), government spending (G_t) and output Y_t. The first column refers to a fiscal surprise which we'll use later.
We're going to concentrate of the sample that Blanchard and Perotti use, 1960 - 1997. Blanchard and Perotti include three deterministic trends in their VAR. To keep things simple, we're going to pretreat–i.e., remove the trends before estimation–the data. *(Why might this be a bad idea?)* The three trends are: a linear trend, a quadratic trend, and a dummy observation for 1975:Q2, when there was a large tax cut.
(In the provided example file, fill in "Code Exercise: Part 1").

Run a reduced-form VAR

Let the detrended data be denoted by $y_t = [t_t, g_t, y_t]'$. Posit that the dynamics of this model follow a VAR(4):

$$y_t = x_t'\Phi + u_t, \quad u_t \sim N(0, \Sigma). \tag{13}$$

Run a VAR(4) on your detrended data. In particular, obtain maximum likelihood estimates, $\hat{\Phi}$, and $\hat{\Sigma}$.

Go from a reduced form to a structural model.

As we discussed in class, the reduced form model does not allow us to conduct structural inference. Blanchard and Perotti propose to identify the structural in the following way. Consider the VAR residuals $u_t = [u_t^t, u_t^g, u_t^y]$. You can think of these as the components of the VAR observables that are not predetermined–the interesting part. Blanchard and Perotti write the relationship between the reduced form residuals and the structural shocks in the following system:

$$u_t^t = a_1 u_t^y + a_2 \sigma_g e_t^g + \sigma_t e_t^t \tag{14}$$

$$u_t^g = b_1 u_t^y + b_2 \sigma_t e_t^t + \sigma_g e_t^g \tag{15}$$

$$u_t^y = c_1 u_t^t + c_2 u_t^g + \sigma_y e_t^y. \tag{16}$$

Blanchard and Perotti make two assumptions in order to solve the identifcation problem. Specifically, they assume particular values for a_1, a_2, and b_2. *What are the values that they assume? Why?*

Write the above the set of equations in matrix notion:

$$F_u u_t = F_\epsilon \epsilon_t. \tag{17}$$

What is the relationshp between F_u, F_ϵ, and Σ? With the Blanchard and Perotti coefficient assumptions, what is the number of free parameters to estimated, and what are the number of equations?

Next, write a `matlab` function that takes arguments $a_1, a_2, b_1, b_2, b_3, \sigma_t, \sigma_g, \sigma_y$, and Σ. Using the relationship derived above, compute the difference between the covariance matrix implied by F_u and F_ϵ and Σ. *Complete the code in objective.m.*

Now consider the structural representation:

$$y_t' A_0 = x_t' A_+ + \epsilon_t', \quad \epsilon_t N(0, I). \tag{18}$$

What is the relationship between A_0 and (F_u, F_ϵ)? Using your results from `fsolve`, construct A_0. With your estimates of A_0 and Φ, construct the impulse response function to an (negative) tax shock. To interpret it as a multiplier–dollar change in GDP as the ratio of dollar change in tax revenue)–rescale the shock so that it's a negative 1\shock, by the average ratio of federal tax revenues to GDP of 17.5%. *Plot the impulse response, and compare it to Blanchard and Perotti.*

A Bayesian Approach

We previously did everything at the MLE. Now, let's add uncertainty using a Bayesian SVAR. Our goal will be to construct draws $\{\Phi^i, \Sigma^i\}_{i=1}^{n_{sim}}$ from the posterior distribution of the reduced form VAR parameters when we use a Minnesota Prior.

We're going to use code from the following handbook chapter to construct the dummy observations for the Minnesota Prior.

- Del Negro, M., & Schorfheide, F. (2011). Bayesian Macroeconometrics. In H. v. Dijk, G. Koop, & J. Geweke (Eds.), *Handbook of Bayesian Econometrics* (pp. 293–389). : Oxford University Press.

In the code, pick some values for hyper parameters of the Minnesota Prior. Next we're going to construct our posterior sampler. Make sure your actual Y and X matrices are called `YYact` and `XXact`. Then produce `nsim` draws from the posterior using the code in `ps3.m`.

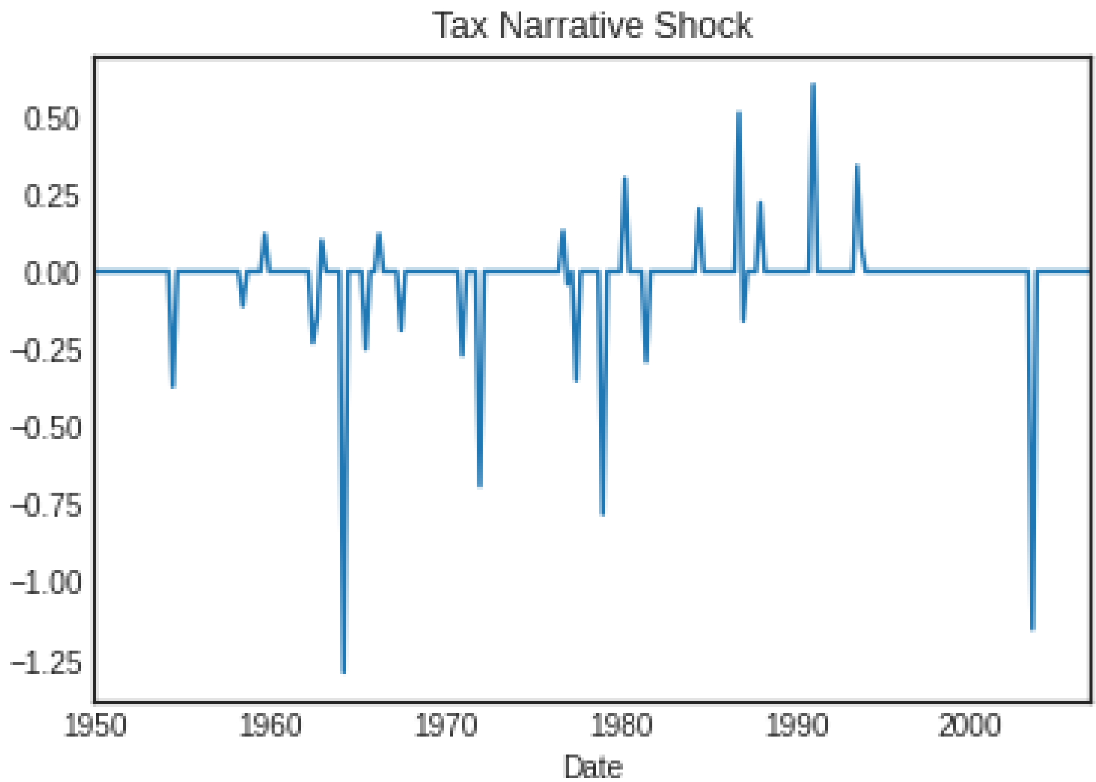
For each draw in Φ^i, Σ^i*, construct the structural impulse response to the tax shock, and plot the median an 90% pointwise probability bands.*

Bonus: Put a prior distribution of a_1, a_2, and b_1–Why are they not updated by the data?–and redo the exercise.

Identification using a proxy

Consider identifing the effects of a tax shock using a proxy. Let's look at the "Tax Narrative" shock constructed by Mertens and Ravn.

```
<ipython-input-6-3ec69544cf50>:4: UserWarning: Could not infer format, so each element will be p
  mr = p.read_csv('mertens_ravn.csv',parse_dates=True,index_col=['Date'])#,sep='\t',index_col=['
```



Let's use the Proxy SVAR approach to estimate the impulse response to a tax shock. Using the maximum likelihood estimates, $\hat{\Phi}, \hat{\Sigma}$, identify the tax shock effect using the proxy–that is, assume that our proxy is a noisy measure of the true tax shock. *Is this what Mertens and Ravn assume?* Use the file `mr.m` to construct the Mertens and Ravn matrix B. *What is the relationship between* A_0 *and* B*?*

Now plot IRF. Is different than your previous one? How? What might be some issues with this identification strategy?

As a final exercise, rather than use the Mertens and Ravn algorithm, simply place the tax shock in the VAR. Using a cholesky identification scheme, can you construct the impulse response to a tax shock? (Where should the series be ordered in the VAR? Why?)